

Stochastic Control and Differential Games with Path-Dependent Controls

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Abstract

In this paper we consider the functional Itô calculus framework to find a path-dependent version of the Hamilton-Jacobi-Bellman equation for stochastic control problems with path-dependence in the controls. We also prove a Dynamic Programming Principle for such problems. We apply our results to path-dependence of the delay type. We further study Stochastic Differential Games in this context.

1 Introduction

Stochastic optimization problems appear naturally in various areas of applications. Portfolio allocation, investment-consumption utility maximization, hedging in incomplete markets and real options are some important examples in Finance and Economics. See for instance, Carmona [2016] and Pham [2009]. The standard case deals with a controlled diffusion

$$\begin{cases} dx_s^{t,y,A_T} = b(s, x_s, \alpha_s)ds + \sigma(s, x_s, \alpha_s)dw_s, & \text{if } s > t, \\ x_t^{t,y,A_T} = y, \end{cases}$$

where $A_T = (\alpha_t)_{t \in [0,T]}$ is an admissible control, and a cost functional

$$J(t, y, A_T) = \mathbb{E} \left[g(x_T^{t,y,A_T}) + \int_t^T f(s, x_s^{t,y,A_T}, \alpha_s) ds \right],$$

with g and f suitable functions. The quantity of interest here is the value function:

$$V(t, y) = \inf_{A_T} J(t, y, A_T).$$

Differently from the usual theory of control, we are denoting the control as A_T instead of α . This notation is consistent with the notation of the functional Itô calculus, as we comment in Section 1.1. Moreover, it makes it explicit the time horizon on which the control is being considered.

Two very important results on Stochastic Control are the Dynamic Programming Principle (DPP) and the Verification Theorem for the related Hamilton-Jacobi-Bellman (HJB) equation.

The main contribution of our paper is to extend the DPP and the HJB to controlled diffusion and cost functional that depend on the path of the control α . The main example to have in mind is the delayed diffusion

$$dx_t^{A_T} = (\alpha_t - \alpha_{t-\tau})dt + \sigma dw_t,$$

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for a fixed τ . The stochastic control theory has been extended to consider path-dependence in the state variable x , see for example Fournié [2010]. This generalization is fundamentally different from the one pursued in our paper, which will become clear in the sections to follow.

Path-dependent controls are still incipient in theory and applications of stochastic control and differential games. This is very likely related to the lack of theoretical tools to deal with such objects in an appropriate way. We hope this work will provide a useful framework.

For example, in Gozzi and Marinelli [2006] and Gozzi and Masiero [2015], the authors considered a class of problems that exhibit a particular type of path-dependence in the control, namely delayed controls. The method implemented there is the classical infinite dimensional analysis and they derived an infinite dimensional HJB equation. However, their method is strongly related to the delay-type of path-dependence. This theory was recently applied to stochastic games and systemic risk in Carmona et al. [2016].

Our approach uses the functional Itô calculus framework, introduced by Bruno Dupire in the seminal paper Dupire [2009], which allows us to consider more general path-dependence structures. Although our method could be also seen as an infinite dimensional analysis, it is rather different than the one applied in Gozzi and Marinelli [2006] and Gozzi and Masiero [2015]. Our method delivers a simpler HJB equation that can be applied to virtually any path-dependence structure in the control. Our assumptions are mainly related to the well-posedness of the optimal control problem (smoothness, measurability and integrability). Functional Itô calculus was applied to the stochastic control problem of portfolio optimization with bounded memory in Pang and Hussain [2015]. However, this reference does not deal with path-dependence in the control, only on the state of the system. Furthermore, this theory was applied in stochastic differential games in the context of zero-sum games in Pham and Zhang [2014].

The structure of the paper is as follows. We finish this introduction with the main definitions and results of functional Itô calculus. In Section 2, we introduce the problem we are considering and we derive the main results of our work: the DPP in Theorem 2.1 and the Verification Theorem for the path-dependent HJB equation in Theorem 2.2. An example is analyzed in Section 2.2. Additionally, in Section 3, we briefly study stochastic differential games with path-dependent actions.

1.1 A Crash Course in Functional Itô Calculus

The important notions of the functional Itô calculus framework will be introduced in this section. For more details and results, we forward the reader to Cont and Fournié [2010a,b, 2013], Dupire [2009], Ekren et al. [2014a,b], Saporito [2014].

We start by fixing a time horizon $T > 0$. Denote Λ_t^d the space of càdlàg paths in $[0, t]$ taking values in \mathbb{R}^d and define $\Lambda^d = \bigcup_{t \in [0, T]} \Lambda_t^d$ and $\Lambda^{d \times k} = \bigcup_{t \in [0, T]} \Lambda_t^d \times \Lambda_t^k$. Elements of $\Lambda^{d \times k}$ are two paths taking values in \mathbb{R}^d and \mathbb{R}^k , respectively, with the same time interval as domain. When it is not necessary to distinguish the dimensions of these space, we will use the notation Λ .

Capital letters will denote elements of Λ (i.e. paths) and lowercase letters will denote spot value of paths. In symbols, $Y_t \in \Lambda$ means $Y_t \in \Lambda_t$ and $y_s = Y_t(s)$, for $s \leq t$.

A functional is any function $f : \Lambda \rightarrow \mathbb{R}$. For such objects, we define the functional derivatives, when these limits exist:

$$(1) \quad \Delta_t f(Y_t) = \lim_{\delta t \rightarrow 0^+} \frac{f(Y_{t, \delta t}) - f(Y_t)}{\delta t},$$

$$(2) \quad \Delta_x f(Y_t) = \lim_{h \rightarrow 0} \frac{f(Y_t^h) - f(Y_t)}{h}.$$

where

$$Y_{t,\delta t}(u) = \begin{cases} y_u, & \text{if } 0 \leq u \leq t, \\ y_t, & \text{if } t \leq u \leq t + \delta t, \end{cases}$$

$$Y_t^h(u) = \begin{cases} y_u, & \text{if } 0 \leq u < t, \\ y_t + h, & \text{if } u = t, \end{cases}$$

see Figures 1 and 2. In the case when the path Y_t lies in a multidimensional space, the path deformations above are understood as follows: the flat extension is applied to all dimension jointly and equally and the bump is applied to each dimension individually.

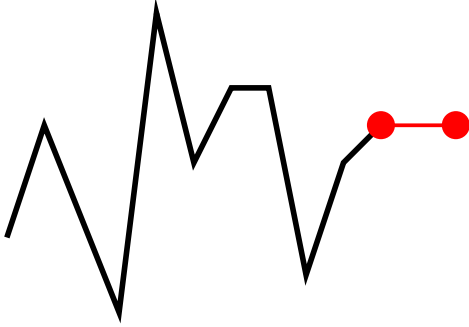


Figure 1: Flat extension of a path.

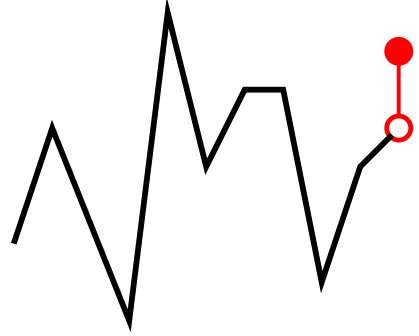


Figure 2: Bumped path.

We consider here continuity in Λ as the usual continuity in metric spaces with respect to the metric:

$$d_\Lambda(Y_t, Z_s) = \|Y_{t,s-t} - Z_s\|_\infty + |s - t|,$$

where, without loss of generality, we assume $s \leq t$, and

$$\|Y_t\|_\infty = \sup_{u \in [0, t]} |y_u|.$$

The norm $|\cdot|$ is the usual Euclidean norm in the appropriate space, depending on the dimension of the path being considered.

Moreover, we say a functional f is *boundedness-preserving* if, for every compact set $K \subset \mathbb{R}$, there exists a constant C such that $|f(Y_t)| \leq C$, for every path Y_t satisfying $Y_t([0, t]) = \{y \in \mathbb{R} ; Y_t(s) = y \text{ for some } s \in [0, t]\} \subset K$.

A functional $f : \Lambda \rightarrow \mathbb{R}$ is said to belong to $\mathbb{C}^{1,2}$ if it is Λ -continuous, boundedness-preserving and it has Λ -continuous, boundedness-preserving derivatives $\Delta_t f$, $\Delta_x f$ and $\Delta_{xx} f$. Here, clearly, $\Delta_{xx} = \Delta_x \Delta_x$.

The Itô formula can be generalized to this framework. The proof can be found in Dupire [2009]. We start by fixing a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 1.1 (Functional Itô Formula; Dupire [2009]). *Let x be a continuous semimartingale and $f \in \mathbb{C}^{1,2}$. Then, for any $t \in [0, T]$,*

$$f(X_t) = f(X_0) + \int_0^t \Delta_t f(X_s) ds + \int_0^t \Delta_x f(X_s) dx_s + \frac{1}{2} \int_0^t \Delta_{xx} f(X_s) d\langle x \rangle_s \quad \mathbb{P}\text{-a.s.}$$

2 Stochastic Control with Path-Dependent Controls

We suggest the reader to always keep this example in mind:

$$dx_t^{A_T} = (\alpha_t - \alpha_{t-\tau})dt + \sigma dw_t.$$

Consider a d -dimensional Brownian motion $(w_t)_{t \in [0, T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and define $(\mathcal{F}_t)_{t \in [0, T]}$ the augmented natural filtration of w . The set of admissible controls $\mathbb{A}(\mathcal{F}_t)$, or just \mathbb{A} , is the space of \mathcal{F}_t -progressively measurable processes in $L^2(\Omega \times [0, T])$ taking values in $\mathcal{A} \subset \mathbb{R}^k$. Additional restrictions will be assumed for \mathbb{A} .

We will use following the notation: $A_T = (\alpha_t)_{t \in [0, T]}$, i.e. the path of the control $\alpha \in \mathbb{A}$,

$$A_{t-}(s) = \begin{cases} \alpha_s, & \text{if } s < t, \\ \lim_{s \rightarrow t-} \alpha_s, & \text{if } s = t, \end{cases}$$

and

$$(3) \quad (Z_t \otimes A_T)(s) = \begin{cases} z_s, & \text{if } s < t, \\ \alpha_s, & \text{if } s \geq t. \end{cases}$$

The path $Z_t \otimes A_T$ is equal Z up to time t (excluding it) and then follows the control α .

We will consider the following path-dependent controlled diffusion dynamics for x :

$$(4) \quad \begin{cases} dx_s^{Y_t, A_T} = b(X_s^{Y_t, A_T}, A_{s-}, \alpha_s)ds + \sigma(X_s^{Y_t, A_T}, A_{s-}, \alpha_s)dw_s, & \text{if } s > t, \\ X_t^{Y_t, A_T} = Y_t, \end{cases}$$

where $b : \Lambda^{d \times k} \times \mathcal{A} \rightarrow \mathbb{R}^d$ and $\sigma : \Lambda^{d \times k} \times \mathcal{A} \rightarrow \mathbb{S}^d$, with \mathbb{S}^d denoting the space of symmetric $d \times d$ matrices. Notice that we are allowing for path-dependence on the state system, x , and on the control α . Writing A_{s-} and α_s , we are separating and making explicit the dependence with respect to the last value of the control. To guarantee existence and uniqueness of strong solutions, we assume there exists $K > 0$ such that

$$\begin{cases} \|b(Y_s, Z_{s-}, z_s) - b(Y'_s, Z_{s-}, z_s)\| \leq K \|Y_s - Y'_s\|_\infty, \\ \|\sigma(Y_s, Z_{s-}, z_s) - \sigma(Y'_s, Z_{s-}, z_s)\| \leq K \|Y_s - Y'_s\|_\infty, \\ \|b(Y_s, Z_{s-}, z_s)\| + \|\sigma(Y_s, Z_{s-}, z_s)\| \leq K(1 + |s| + \|Y_s\|_\infty), \end{cases}$$

for all $s \geq t$, $(Y_s, Z_s), (Y'_s, Z_s) \in \Lambda^{d \times k}$.

Moreover, we consider the following class of cost functionals $J : \Lambda^d \times \mathbb{A} \rightarrow \mathbb{R}$:

$$(5) \quad J(Y_t, A_T) = \mathbb{E} \left[g(X_T^{Y_t, A_T}) + \int_t^T f(X_s^{Y_t, A_T}, A_{s-}, \alpha_s) ds \right],$$

where $g : \Lambda_T^d \rightarrow \mathbb{R}$ and $f : \Lambda^{d \times k} \times \mathcal{A} \rightarrow \mathbb{R}$ satisfy certain measurability and integrability conditions. Notice that $J(Y_T, A_T) = g(Y_T)$. We additionally assume that the admissible controls in \mathbb{A} satisfy certain straightforward integrability conditions depending on the functionals b, σ and f so that Equations (4) and (5) are well-defined.

We define then the value function $V : \Lambda^{d \times k} \rightarrow \mathbb{R}$:

$$V(Y_t, Z_t) = \inf_{A_T \in \mathbb{A}} J(Y_t, Z_t \otimes A_T).$$

Theorem 2.1 (Dynamic Programming Principle (DPP)). *For any $u \in [t, T]$,*

$$V(Y_t, Z_t) = \inf_{A_T \in \mathbb{A}} \mathbb{E} \left[V(X_u^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_u) + \int_t^u f(X_s^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_{s-}, \alpha_s) ds \right]$$

Proof. The proof is the same as in the path-independent case, since all the coefficients are still adapted. We follow the structure of the proof in Pham [2009].

Firstly, notice that, for any $A_T \in \mathbb{A}$ and $t \leq u \leq s \leq T$,

$$X_s^{Y_t, A_T} = X_s^{X_u^{Y_t, A_T}, A_T}.$$

Then,

$$J(Y_T, A_T) = \mathbb{E} \left[g(X_T^{X_u^{Y_t, A_T}, A_T}) + \int_t^u f(X_s^{Y_t, A_T}, A_{s-}, \alpha_s) ds + \int_u^T f(X_s^{X_u^{Y_t, A_T}, A_T}, A_{s-}, \alpha_s) ds \right],$$

and conditioning on the path $X_u^{Y_t, A_T}$, we find

$$(6) \quad J(Y_T, A_T) = \mathbb{E} \left[\int_t^u f(X_s^{Y_t, A_T}, A_{s-}, \alpha_s) ds + J(X_u^{Y_t, A_T}, A_T) \right].$$

From this and choosing the control A_T to be $Z_t \otimes A_T$, it is clear that

$$J(Y_T, Z_t \otimes A_T) \geq \mathbb{E} \left[V(X_u^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_u) + \int_t^u f(X_s^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_{s-}, \alpha_s) ds \right].$$

Taking the infimum with respect to $A_T \in \mathbb{A}$, we find

$$V(Y_t, Z_t) \geq \inf_{A_T \in \mathbb{A}} \mathbb{E} \left[V(X_u^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_u) + \int_t^u f(X_s^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_{s-}, \alpha_s) ds \right].$$

To prove the opposite inequality, fix $A_T \in \mathbb{A}$ and $u \in [t, T]$. Then, for any $\varepsilon > 0$, there exists $A_T^\varepsilon \in \mathbb{A}$ such that

$$V(X_u^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_u) + \varepsilon \geq J(X_u^{Y_t, (Z_t \otimes A_T)_u \otimes A_T^\varepsilon}, (Z_t \otimes A_T)_u \otimes A_T^\varepsilon)$$

It can be shown by the measurable selection theorem that $\hat{A}_T = A_u \otimes A_T^\varepsilon$ belongs to \mathbb{A} (i.e. it is progressively measurable). Since $Z_t \otimes \hat{A}_T = (Z_t \otimes A_T)_u \otimes A_T^\varepsilon$, by Equation (6), we find

$$\begin{aligned} V(Y_t, Z_t) &\leq J(Y_t, Z_t \otimes \hat{A}_T) \\ &= \mathbb{E} \left[\int_t^u f(X_s^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_{s-}, \alpha_s) ds + J(X_u^{Y_t, Z_t \otimes \hat{A}_T}, Z_t \otimes \hat{A}_T) \right] \\ &\leq \mathbb{E} \left[\int_t^u f(X_s^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_{s-}, \alpha_s) ds + V(X_u^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_u) \right] + \varepsilon, \end{aligned}$$

which implies, by the fact $A_T \in \mathbb{A}$ and $\varepsilon > 0$ are arbitrary, that

$$V(Y_t, Z_t) \leq \inf_{A_T \in \mathbb{A}} \mathbb{E} \left[V(X_u^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_u) + \int_t^u f(X_s^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_{s-}, \alpha_s) ds \right],$$

from where the final result follows. \square

2.1 The Path-Dependent Hamilton-Jacobi-Bellman Equation

In this section we will state the HJB equation related to our control problem and also prove a verification theorem for such equation. In the framework of the functional Itô calculus, this type of equation is called Path-Dependent Partial Differential Equation, PPDE. See for example, Ekren et al. [2014a] and Ekren et al. [2014b].

We start by defining $H : \Lambda^{d \times k} \times \mathbb{R}^d \times \mathbb{S}^d \times \mathcal{A} \longrightarrow \mathbb{R}$:

$$H(Y_t, Z_t, p, \gamma, \alpha) = \frac{1}{2} \sigma \sigma^T(Y_t, Z_{t-}, \alpha) : \gamma + b(Y_t, Z_{t-}, \alpha) \cdot p + f(Y_t, Z_{t-}, \alpha)$$

and the Hamiltonian $\hat{H} : \Lambda^{d \times k} \times \mathbb{R}^d \times \mathbb{S}^d \longrightarrow \mathbb{R}$:

$$\hat{H}(Y_t, Z_t, p, \gamma) = \inf_{\alpha \in \mathcal{A}} H(Y_t, Z_t, p, \gamma, \alpha).$$

The notation \cdot and $:$ mean

$$p \cdot q = \sum_{i=1}^d p_i q_i \text{ and } \gamma : \phi = \text{trace}(\gamma \phi),$$

where $p, q \in \mathbb{R}^d$ and $\gamma, \phi \in \mathbb{S}^d$.

As we will conclude, the HJB equation in this case is given by the following PPDE:

$$(7) \quad \begin{cases} \Delta_t V(Y_t, Z_t) + \hat{H}(Y_t, Z_t, \Delta_x V(Y_t, Z_t), \Delta_{xx} V(Y_t, Z_t)) = 0, \\ V(Y_T, Z_T) = g(Y_T), \end{cases}$$

for any $Z_t \in \Lambda$. Here, the time derivative Δ_t is with respect to both variable Y and Z :

$$\Delta_t V(Y_t, Z_t) = \lim_{\delta t \rightarrow 0^+} \frac{V(Y_{t, \delta t}, Z_{t, \delta t}) - V(Y_t, Z_t)}{\delta t}.$$

Theorem 2.2 (Verification Theorem). *Suppose $V \in \mathbb{C}^{1,2}$ solves the HJB equation (7). Under mild integrability conditions,*

$$V(Y_t, Z_t) \leq J(Y_t, Z_t \otimes A_T),$$

for any $A_T \in \mathbb{A}$. Moreover, if there exists $A_T^* \in \mathbb{A}$ such that

$$(8) \quad \begin{aligned} & \hat{H}(X_u^{Y_t, Z_t \otimes A_T^*}, (Z_t \otimes A_T^*)_{u-}, \Delta_x V, \Delta_{xx} V) \\ & = H(X_u^{Y_t, Z_t \otimes A_T^*}, (Z_t \otimes A_T^*)_{u-}, \Delta_x V, \Delta_{xx} V, \alpha_u^*), \end{aligned}$$

then $V(Y_t, Z_t) = J(Y_t, Z_t \otimes A_T^*)$. All the functional derivatives in (8) are computed at $(X_u^{Y_t, Z_t \otimes A_T^*}, Z_t)$.

Proof. It is important to notice that $V(Y_t, Z_{t-}) = V(Y_t, Z_t)$, by the definition of \otimes given in (3). In particular, $V(Y_t, Z_t^h) = V(Y_t, Z_t)$. Denoting the functional derivative with respect to Y and Z by Δ_x and Δ_α , respectively, we conclude $\Delta_\alpha V(Y_t, Z_t) = 0$, $\Delta_{\alpha\alpha} V(Y_t, Z_t) = 0$ and $\Delta_{x\alpha} V(Y_t, Z_t) = 0$. Hence, the dynamics of the control A_T will not impact the computations that follow. This is similar to what Cont and Fournié assumed to consider functionals depending on the quadratic variation, see Cont and Fournié [2010a].

Let us apply the Functional Itô Formula, Theorem 1.1, to $V(X_s^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_s)$, for fixed $A_T \in \mathbb{A}$. Notice that the path Z is frozen and that we are considering the control $Z_t \otimes A_T$, which means we follow the path Z_t as the control up to time t (excluding it) and then A_T from t to T . Moreover, since the

functional derivatives of V with respect to the control α are zero, it is not necessary to consider the dynamics of the control α . Furthermore, the time derivative is with respect to both variables. In the computation that follows we suppress the superscript of $X_s^{Y_t, Z_t \otimes A_T}$ for a cleaner exposition

$$\begin{aligned}
V(X_s, (Z_t \otimes A_T)_s) &= V(Y_t, Z_t) + \int_t^s \Delta_t V(X_u, (Z_t \otimes A_T)_u) du \\
&+ \int_t^s \Delta_x V(X_u, (Z_t \otimes A_T)_u) \cdot b(X_u, (Z_t \otimes A_T)_{u-}, \alpha_u) du \\
&+ \int_t^s \Delta_x V(X_u, (Z_t \otimes A_T)_u) \cdot \sigma(X_u, (Z_t \otimes A_T)_{u-}, \alpha_u) dw_u \\
&+ \frac{1}{2} \int_t^s \Delta_{xx} V(X_u, (Z_t \otimes A_T)_u) : \sigma \sigma^T(X_u, (Z_t \otimes A_T)_{u-}, \alpha_u) du \\
&= V(Y_t, Z_t) + \int_t^s \Delta_t V(X_u, (Z_t \otimes A_T)_u) du + \int_t^s H(X_u, (Z_t \otimes A_T)_{u-}, \Delta_x V, \Delta_{xx} V, \alpha_u) du \\
&+ \int_t^s \Delta_x V(X_u, (Z_t \otimes A_T)_u) \cdot \sigma(X_u, (Z_t \otimes A_T)_{u-}, \alpha_u) dw_u - \int_t^s f(X_u, (Z_t \otimes A_T)_{u-}, \alpha_u) du \\
&\geq V(Y_t, Z_t) + \int_t^s \Delta_t V(X_u, (Z_t \otimes A_T)_u) du + \int_t^s \widehat{H}(X_u, (Z_t \otimes A_T)_{u-}, \Delta_x V, \Delta_{xx} V) du \\
&+ \int_t^s \Delta_x V(X_u, (Z_t \otimes A_T)_u) \cdot \sigma(X_u, (Z_t \otimes A_T)_{u-}, \alpha_u) dw_u - \int_t^s f(X_u, (Z_t \otimes A_T)_{u-}, \alpha_u) du \\
&= V(Y_t, Z_t) + \int_t^s \Delta_x V(X_u, (Z_t \otimes A_T)_u) \cdot \sigma(X_u, (Z_t \otimes A_T)_{u-}, \alpha_u) dw_u - \int_t^s f(X_u, (Z_t \otimes A_T)_{u-}, \alpha_u) du.
\end{aligned}$$

Under integrability conditions and applying localization techniques, the Itô integral above is a martingale. Therefore, taking expectation on both sides, we find:

$$V(Y_t, Z_t) \leq \mathbb{E} \left[V(X_s^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_s) + \int_t^s f(X_u^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_{u-}, \alpha_u) du \right].$$

Letting $s \nearrow T$:

$$V(Y_t, Z_t) \leq \mathbb{E} \left[g(X_T^{Y_t, Z_t \otimes A_T}) + \int_t^T f(X_u^{Y_t, Z_t \otimes A_T}, (Z_t \otimes A_T)_{u-}, \alpha_u) du \right] = J(Y_t, Z_t \otimes A_T),$$

as desired

Taking the control A_T^* satisfying Equation (8), we find

$$V(Y_t, Z_t) = \mathbb{E} \left[V(X_s^{Y_t, Z_t \otimes A_T^*}, (Z_t \otimes A_T^*)_s) + \int_t^s f(X_u^{Y_t, Z_t \otimes A_T^*}, (Z_t \otimes A_T^*)_{u-}, \alpha_u^*) du \right].$$

Therefore, letting $s \nearrow T$ as well, we find $V(Y_t, Z_t) = J(Y_t, Z_t \otimes A_T^*)$. \square

Remark 2.3. It is obvious that if the dynamics of x and the functionals g and f are path-independent in the state variable and control, we find the classical HJB equation. Moreover, if the path-dependence is only in the control, meaning that

$$h(Y_t, Z_t, \alpha) = h(t, y_t, Z_t, \alpha) \text{ and } g(Y_T) = g(y_T),$$

for $h = b, \sigma, f$, the path-dependent HJB Equation (7) becomes

$$(9) \quad \begin{cases} \Delta_t V(t, y, Z_t) + \widehat{H}(t, y_t, Z_t, \partial_x V(t, y, Z_t), \partial_{xx} V(t, y, Z_t)) = 0, \\ V(T, y, Z_T) = g(y), \end{cases}$$

where ∂_x is the usual derivative with respect to the state variable and

$$\widehat{H}(t, y, Z_t, p, \gamma) = \inf_{\alpha \in \mathcal{A}} \left\{ \frac{1}{2} \sigma \sigma^T(t, y, Z_{t-}, \alpha) : \gamma + b(t, y, Z_{t-}, \alpha) \cdot p + f(t, y, Z_{t-}, \alpha) \right\}.$$

It is worth noticing that Δ_t is still a functional derivative. More precisely, it is giving by

$$\Delta_t V(t, y, Z_t) = \lim_{\delta t \rightarrow 0^+} \frac{V(t + \delta t, y, Z_{t, \delta t}) - V(t, y, Z_t)}{\delta t}.$$

2.2 Example: Delayed Control

We will exemplify the results derived in the section above, mainly the path-dependent HJB equation, by considering the simple linear-quadratic example

$$\begin{aligned} b(t, y, Z_t, \alpha) &= \alpha - z_{t-\tau}, \quad \sigma(t, y, Z_t, \alpha) = \sigma, \\ f(t, y, Z_t, \alpha) &= \frac{\alpha^2}{2} + \frac{\varepsilon}{2} y^2 \text{ and } g(y) = c \frac{y^2}{2}. \end{aligned}$$

This example allows for complete computation of the solution (up to computing the solution of a system of PDEs). Indeed, notice that

$$H(t, y, Z_t, p, \gamma, \alpha) = \frac{\sigma^2}{2} \gamma + (\alpha - z_{t-\tau}) p + \frac{\alpha^2}{2} + \frac{\varepsilon}{2} y^2,$$

which implies that $\widehat{H}(t, y, Z_t, p, \gamma) = H(t, y, Z_t, p, \gamma, \widehat{\alpha})$, where $\widehat{\alpha} = -p$. The HJB equation in this example becomes:

$$(10) \quad \begin{cases} \Delta_t V(t, y, Z_t) + \frac{\sigma^2}{2} \partial_{xx} V(t, y, Z_t) - \frac{1}{2} (\partial_x V(t, y, Z_t))^2 - z_{t-\tau} \partial_x V(t, y, Z_t) + \frac{\varepsilon}{2} y^2 = 0, \\ V(T, y, Z_T) = c \frac{y_T^2}{2}. \end{cases}$$

We consider the following ansatz (as it was considered in Gozzi and Marinelli [2006]):

$$V(t, y, Z_t) = F_0(t) \frac{y^2}{2} + y \int_{t-\tau}^t F_1(t, \theta - t) z_\theta d\theta + \int_{t-\tau}^t \int_{t-\tau}^t F_2(t, \theta_1 - t, \theta_2 - t) z_{\theta_1} z_{\theta_2} d\theta_1 d\theta_2 + F_3(t).$$

We can compute these derivatives explicitly. Δ_t would be more complicated, but for this ansatz, it may be verified that it is equivalent to taking derivative with respect to t :

$$\begin{aligned} \partial_x V &= F_0(t) y + \int_{t-\tau}^t F_1(t, \theta - t) z_\theta d\theta, \\ \partial_{xx} V &= F_0(t), \\ \Delta_t V &= F_0'(t) \frac{y^2}{2} + y(F_1(t, 0) z_t - F_1(t, -\tau) z_{t-\tau}) + y \int_{t-\tau}^t \left(\frac{\partial F_1}{\partial t} - \frac{\partial F_1}{\partial \theta} \right) (t, \theta - t) z_\theta d\theta \\ &\quad + z_t \int_{t-\tau}^t F_2(t, \theta - t, 0) z_\theta d\theta - z_{t-\tau} \int_{t-\tau}^t F_2(t, \theta - t, -\tau) z_\theta d\theta \\ &\quad + z_t \int_{t-\tau}^t F_2(t, 0, \theta - t) z_\theta d\theta - z_{t-\tau} \int_{t-\tau}^t F_2(t, -\tau, \theta - t) z_\theta d\theta \\ &\quad + \int_{t-\tau}^t \int_{t-\tau}^t \left(\frac{\partial F_2}{\partial t} - \frac{\partial F_2}{\partial \theta_1} - \frac{\partial F_2}{\partial \theta_2} \right) (t, \theta_1 - t, \theta_2 - t) z_{\theta_1} z_{\theta_2} d\theta_1 d\theta_2 + F_3'(t). \end{aligned}$$

Combining all derivatives into HJB Equation (10)

$$\left\{ \begin{array}{l} F'_0(t) - F_0^2(t) + \varepsilon = 0, \\ F_1(t, 0)z_t - (F_1(t, -\tau) + F_0(t))z_{t-\tau} + \int_{t-\tau}^t \left(\frac{\partial F_1}{\partial t} - \frac{\partial F_1}{\partial \theta} - F_0(t)F_1 \right) (t, \theta - t)z_\theta d\theta = 0, \\ F'_3(t) + \frac{\sigma^2}{2}F_0(t) + z_t \int_{t-\tau}^t (F_2(t, \theta, 0) + F_2(t, 0, \theta))z_\theta d\theta \\ - z_{t-\tau} \int_{t-\tau}^t (F_2(t, \theta, -\tau) + F_2(t, -\tau, \theta) + F_1(t, \theta - t))z_\theta d\theta \\ + \int_{t-\tau}^t \int_{t-\tau}^t \left(\left(\frac{\partial F_2}{\partial t} - \frac{\partial F_2}{\partial \theta_1} - \frac{\partial F_2}{\partial \theta_2} \right) (t, \theta_1 - t, \theta_2 - t) - \frac{1}{2}F_1(t, \theta_1 - t)F_1(t, \theta_2 - t) \right) z_{\theta_1}z_{\theta_2}d\theta_1d\theta_2 = 0, \end{array} \right.$$

with the following boundary conditions:

$$\left\{ \begin{array}{l} F_0(T) = c, \\ F_1(T, \theta - T) = 0, \quad \forall \theta \in (T - \tau, T) \\ F_2(T, \theta_1 - T, \theta_2 - T) = 0, \quad \forall \theta_1, \theta_2 \in (T - \tau, T) \\ F_3(T) = 0. \end{array} \right.$$

Since the realized control Z_t is arbitrary, we find that, for any $\theta, \theta_1, \theta_2 \in (t - \tau, t)$,

$$(11) \quad \left\{ \begin{array}{l} F'_0(t) - F_0^2(t) + \varepsilon = 0, \\ F_0(T) = c, \end{array} \right.$$

$$(12) \quad \left\{ \begin{array}{l} \left(\frac{\partial F_1}{\partial t} - \frac{\partial F_1}{\partial \theta} \right) (t, \theta - t) - F_0(t)F_1(t, \theta - t) = 0, \\ F_1(t, 0) = 0, \\ F_1(t, -\tau) = -F_0(t), \\ F_1(T, \theta) = 0, \end{array} \right.$$

$$(13) \quad \left\{ \begin{array}{l} \left(\frac{\partial F_2}{\partial t} - \frac{\partial F_2}{\partial \theta_1} - \frac{\partial F_2}{\partial \theta_2} \right) (t, \theta_1 - t, \theta_2 - t) - \frac{1}{2}F_1(t, \theta_1 - t)F_1(t, \theta_2 - t) = 0, \\ F_2(t, \theta, 0) = -F_2(t, 0, \theta), \\ F_2(t, \theta, -\tau) + F_2(t, -\tau, \theta) + F_1(t, \theta - t) = 0, \end{array} \right.$$

$$(14) \quad \left\{ \begin{array}{l} F'_3(t) + \frac{\sigma^2}{2}F_0(t) = 0, \\ F_3(T) = 0. \end{array} \right.$$

3 Stochastic Differential Games

In this section, we will briefly analyze Stochastic Differential Games. Firstly, we present the general theory relating the game value function and a version of the HJB equation when there is path-dependence in the control. Then, we exemplify the theory using the delayed stochastic differential game proposed in Carmona et al. [2016].

Consider N agents indexed by $i = 1, \dots, N$. These agents will act on a system whose state is described below:

$$\begin{cases} dx_s^{i,Y_t,A_T} = b^i(X_s^{Y_t,A_T}, A_{s-}, \alpha_s)ds + \sigma^i(X_s^{Y_t,A_T}, A_{s-}, \alpha_s)dw_u^i, & \text{if } s > t, \\ X_t^{i,Y_t,A_T} = Y_t^i, \end{cases}$$

for $i = 1, \dots, N$, where w^i is a m_i -dimensional standard Brownian motion, $A_T = (A_T^1, \dots, A_T^N)$ with A_T^i being the k_i -dimensional control chosen by agent i . These Brownian motions could be correlated. Moreover,

$$(b^i, \sigma^i) : \Lambda^{d \times k} \times \mathcal{A}^i \longrightarrow \mathbb{R}^{d_i} \times \mathbb{R}^{d_i \times m_i}$$

where \mathcal{A}^i is the set of actions of the agent i , with $d = d_1 \times \dots \times d_N$ and $k = k_1 \times \dots \times k_N$. We will use the notation $x_s^{Y_t,A_T} = (x_s^{1,Y_t,A_T}, \dots, x_s^{N,Y_t,A_T})$. The set of admissible controls of agent i is denoted by \mathbb{A}^i and $\mathbb{A} = \mathbb{A}^1 \times \dots \times \mathbb{A}^N$.

The agent i chooses its own control α^i to minimize its own objective function:

$$J^i(Y_t, A_T) = \mathbb{E} \left[g^i(X_T^{Y_t,A_T}) + \int_t^T f^i(X_s^{Y_t,A_T}, A_{s-}, \alpha_s) ds \right],$$

where $g^i : \Lambda^d \longrightarrow \mathbb{R}$ and $f^i : \Lambda^{d \times k} \times \mathcal{A} \longrightarrow \mathbb{R}$ are his/hers terminal and running costs. In what follows, we will seek a *closed-loop Nash equilibrium*.

Assuming that the other $N-1$ agents have already chosen their actions, denoted by $A_T^{-i*} = (A_T^{1*}, \dots, A_T^{(i-1)*}, A_T^{(i+1)*}, \dots, A_T^{N*})$, the value function for agent i will be then given by

$$V^i(Y_t, Z_t) = \inf_{A_T^i \in \mathbb{A}^i} J^i(Y_t, Z_t \otimes (A_T^{-i*}, A_T^i)),$$

where $(A_T^{-i*}, A_T^i) = (A_T^{1*}, \dots, A_T^{(i-1)*}, A_T^i, A_T^{(i+1)*}, \dots, A_T^{N*})$. Therefore, under the assumptions of Theorem 2.2, we have a verification theorem for the following HJB Equation

$$\begin{cases} \Delta_t V^i(Y_t, Z_t) + \widehat{H}^i(Y_t, Z_t, \Delta_x V^i(Y_t, Z_t), \Delta_{xx} V^i(Y_t, Z_t)) = 0, \\ V^i(Y_T, Z_T) = g^i(Y_T), \end{cases}$$

where

$$\widehat{H}^i(Y_t, Z_t, p, \gamma) = \inf_{\alpha^i \in \mathcal{A}^i} \left\{ \frac{1}{2} \sigma \sigma^T(Y_t, Z_{t-}, \alpha) : \gamma + b(Y_t, Z_{t-}, \alpha) \cdot p + f^i(Y_t, Z_{t-}, \alpha) \right\}.$$

3.1 Delayed Games

In this section, we will study the model proposed in Carmona et al. [2016], where the authors proposes a stochastic differential game with delay in the control to analyze the systemic risk within a bank system.

Fix $d_i = k_i = m_i = 1$ and

$$b^i(Y_t, Z_t, \alpha) = \alpha^i - z_{t-\tau}^i,$$

$$\begin{aligned}
\sigma^i(Y_t, Z_t, \alpha) &= \sigma, \\
f^i(Y_t, Z_t, \alpha) &= \frac{(\alpha^i)^2}{2} - q\alpha^i(\bar{y}_t - y_t^i) + \frac{\varepsilon}{2}(\bar{y}_t - y_t^i)^2, \\
g^i(Y_T) &= \frac{c}{2}(\bar{y}_T - y_T^i)^2,
\end{aligned}$$

where $\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i$. Assuming that α_j is chosen, for $j \neq i$, the optimal control for the player i is given by $\hat{\alpha}_i = q(\bar{y}_t - y_t^i) - p_i$. Assuming each player is following this strategy and noticing that p_j in the formula for $\hat{\alpha}_j$ should be replaced by $\Delta_{x_j} V^j$ (and not $\Delta_{x_j} V^i$), we find that the Hamiltonians in this case become

$$\begin{aligned}
\widehat{H}^i(Y_t, Z_t, p, \gamma) &= \inf_{\alpha \in \mathcal{A}} \left\{ \frac{\sigma^2}{2} \text{Tr}(\gamma) + (\alpha - z_{t-\tau}) \cdot p + \frac{(\alpha^i)^2}{2} - q\alpha^i(\bar{y}_t - y_t^i) + \frac{\varepsilon}{2}(\bar{y}_t - y_t^i)^2 \right\} \\
&= \frac{\sigma^2}{2} \text{Tr}(\gamma) + (\hat{\alpha} - z_{t-\tau}) \cdot p + \frac{(\hat{\alpha}^i)^2}{2} - q\hat{\alpha}^i(\bar{y}_t - y_t^i) + \frac{\varepsilon}{2}(\bar{y}_t - y_t^i)^2
\end{aligned}$$

The HJB equation turns into

$$\begin{cases} \Delta_t V^i + \sum_{j=1}^N \left(\frac{\sigma^2}{2} \Delta_{x_j x_j} V^i + (q(\bar{y}_t - y_t^j) - \Delta_{x_j} V^j - z_{t-\tau}^j) \Delta_{x_j} V^i \right) \\ + \frac{1}{2} (\Delta_{x_i} V^i)^2 + \frac{1}{2} (\varepsilon - q^2) (\bar{y}_t - y_t^i)^2 = 0, \\ V(Y_T, Z_T) = \frac{c}{2} (\bar{y}_T - y_T^i)^2. \end{cases}$$

Considering the same ansatz as in Carmona et al. [2016],

$$\begin{aligned}
V^i(Y_t, Z_t) &= E_0(t)(\bar{y}_t - y_t^i)^2 - 2(\bar{y}_t - y_t^i) \int_{t-\tau}^t E_1(t, \theta)(\bar{z}_\theta - z_\theta) d\theta \\
&\quad + \int_{t-\tau}^t \int_{t-\tau}^t E_2(t, \theta_1, \theta_2)(\bar{z}_{\theta_1} - z_{\theta_1})(\bar{z}_{\theta_2} - z_{\theta_2}) d\theta_1 d\theta_2 + E_3(t),
\end{aligned}$$

one might readily find the same solution as in the aforesaid reference.

4 Conclusion and Future Research

In this paper, we have studied stochastic control and differential games when there exists path-dependence in the control (or action) of the agent. We have studied the important example of delayed dependence. The framework used was the relatively recent functional Itô calculus, which has been proven to be an excellent tool to deal with complicated path-dependence structures, Jazaerli and Saporito [2013]. Although we have focused on delayed dependence, because of practical importance, there are no major impediments to consider more interesting structures. We hope this work will allow the consideration of these different path-dependence structures in other problems.

Compared to the theory of Gozzi and Marinelli [2006], that deals with just the delay case, the method proposed here allows in principle very general path-dependence in the controls. Moreover, the HJB found here is significantly simpler than the one of the previous reference.

Future research will be conducted to analyze viscosity solutions (existence and uniqueness) of the path-dependent HJB derived here. These equations are called PPDE and have been extensively studied in recent years, see for example Ekren et al. [2014a,b].

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